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Linkage of Cohen–Macaulay modules over a Gorenstein ring

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Abstract

Let R be a Gorenstein complete local ring. We say that finitely generated modules M and N are linked if $\mathrm{Hom}_{R/\underline{\lambda}R}(M, R/\underline{\lambda}R) \cong \Omega^1_{R/\underline{\lambda}R}(N)$, where $\underline{\lambda}$ is a regular sequence contained in both of the annihilators of M and N. We shall show that the Cohen–Macaulay approximation functor gives rise to a map Φ_r from the set of even linkage classes of Cohen–Macaulay modules of codimension r to the set of isomorphism classes of maximal Cohen–Macaulay modules. When $r = 1$, we give a condition for two modules to have the same image under the map Φ_1. If $r = 2$ and if R is a normal domain of dimension two, then we can show that Φ_2 is a surjective map if and only if R is a unique factorization domain. Several explicit computations for hypersurface rings are also given.

MSC: 13C14; 13C40

0. Introduction

Throughout the present paper, let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local ring and we assume that all modules are finitely generated.

The notion of linkage for ideals of R has been introduced by Peskine–Szpiro [6]. Motivated by the work [4] of Herzog and Kühl, we shall extend this notion to modules that are Cohen–Macaulay of high codimension.

To be more precise, let I and J be ideals of R and we recall that I is (algebraically) linked to J via a regular sequence $\underline{\lambda}$ contained in $I \cap J$ if and only if

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$\mathrm{Hom}_{R/\underline{\lambda}R}(R/I, R/\underline{\lambda}R) \cong J/\underline{\lambda}R$. Therefore, it seems natural to define that R-modules M and N are linked via $\underline{\lambda}$ if $\underline{\lambda}M = \underline{\lambda}N = 0$ and $\mathrm{Hom}_{R/\underline{\lambda}R}(M, R/\underline{\lambda}R) \cong \Omega^1_{R/\underline{\lambda}R}(N)$, where $\Omega^1_{R/\underline{\lambda}R}$ is the (first) syzygy functor. Actually we adopt this definition of linkage for Cohen–Macaulay modules M and N (definitions 1.1 and 1.3).

In the case of ideals, there is a useful theory called Rao correspondence [7]. After defining the linkage for Cohen–Macaulay modules, we shall notice that such a correspondence should be reproduced by the Cohen–Macaulay approximation functor in our context.

Recall from [1] that for any R-module M, there is an exact sequence

$$0 \to \mathrm{Y}_R(M) \to \mathrm{X}_R(M) \to M \to 0,$$

where $\mathrm{X}_R(M)$ is a maximal Cohen–Macaulay module and $\mathrm{Y}_R(M)$ is an R-module of finite projective dimension. This construction naturally yields the functor X_R from the category of R-modules to the stable category of maximal Cohen–Macaulay modules, and we call this functor the Cohen–Macaulay approximation functor.

In Corollary 1.6 we shall show that X_R is constant for modules in an even linkage class. Therefore, we can define the map Φ_r from the set of even linkage classes of Cohen–Macaulay modules of codimension $r > 0$ to the set of isomorphism classes of maximal Cohen–Macaulay modules over R (just by sending the even linkage class of M to the isomorphism class of $\mathrm{X}_R(M)$).

The main purpose of this paper is to provide several properties of the map Φ_r for $r > 0$.

In the case $r = 1$, as we shall remark in the beginning of Section 2, Φ_1 is surjective if R is an integral domain, and the condition for two Cohen–Macaulay modules of codimension one to have the same image under Φ_1 is given in Proposition (3.1).

For $r = 2$, we are able to give a condition for Φ_2 to be surjective if R is a normal domain of dimension two. Surprisingly enough, the necessary and sufficient condition for this is that R is a UFD. See Theorem 2.2.

In the last section, as an application, we take a hypersurface ring as R, on which even linkage classes of Cohen–Macaulay modules are comparatively easy to determine.

1. Linkage of modules and the map Φ_r

As in the introduction, we always assume that $(R, \mathfrak{m}, k)$ is a Gorenstein complete local ring of dimension d. We denote the category of finitely generated R-modules by R-mod and denote the category of maximal Cohen–Macaulay modules (resp. the category of Cohen–Macaulay modules of codimension r) as a full subcategory of R-mod by $\mathrm{CM}(R)$ (resp. $\mathrm{CM}^r(R)$). We also denote the stable category by $\underline{\mathrm{CM}}(R)$ (resp. $\underline{\mathrm{CM}}^r(R)$) that is defined in such a way that the objects are the same as that of $\mathrm{CM}(R)$ (resp. $\mathrm{CM}^r(R)$), while the morphisms from M to N are the elements of $\underline{\mathrm{Hom}}_R(M, N) = \mathrm{Hom}_R(M, N)/P(M, N)$ where $P(M, N)$ is the set of morphisms which factor through free R-modules. First we recall the definition of Cohen–Macaulay approximations from the paper [1] of Auslander and Buchweitz. It is shown in [1] that for any $M \in R$-mod, there is an exact sequence

$$0 \to \mathrm{Y}_R(M) \to \mathrm{X}_R(M) \to M \to 0,$$

where $\mathrm{X}_R(M) \in \mathrm{CM}(R)$ and $\mathrm{Y}_R(M)$ is of finite projective dimension. Such a sequence is not unique, but $\mathrm{X}_R(M)$ is known to be unique up to free summand, and hence it gives rise to the functor

$$\mathrm{X}_R : \underline{R\text{-mod}} \to \underline{\mathrm{CM}}(R),$$

which we call the Cohen–Macaulay approximation functor. Let us denote by D_R the R-dual functor $\mathrm{Hom}(\quad, R)$. Note that D_R yields a duality on the category $\mathrm{CM}(R)$. Given an R-module M, we denote the ith syzygy module of M by $\Omega_R^i(M)$ for a non-negative integer i. We should notice that if $i \geq d$, then Ω_R^i gives rise to the functor $\underline{R\text{-mod}} \to \underline{\mathrm{CM}}(R)$. If $M \in \underline{\mathrm{CM}}(R)$, then we can also consider $\Omega_R^i(M)$ even for a negative integer i, which is defined to be $D_R(\Omega_R^{-i}(D_R(M)))$. We call $\Omega_R^i(M)$ the $(-i)$th cosyzygy module of M if $i<0$ and if $M \in \underline{\mathrm{CM}}(R)$. In such a way we get the functor $\Omega_R^i : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R)$ for any integer i. Note that the Cohen–Macaulay approximation functor X_R is just equal to the composite $\Omega_R^{-d} \circ \Omega_R^d$ as a functor from $\underline{R\text{-mod}}$ to $\underline{\mathrm{CM}}(R)$.

Definition 1.1 (*Linkage functor L_R*). We define the functor $L_R : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R)$ by $L_R = D_R \circ \Omega_R^1$.

We should notice from the definition that $L_R^2 \cong \mathrm{id}_{\underline{\mathrm{CM}}(R)}$.

Lemma 1.2. *Let $M \in \underline{\mathrm{CM}}(R)$ and let $\underline{\lambda}$ be a regular sequence in $\mathfrak{m}$. Then we have the isomorphism $L_{R/\underline{\lambda}R}(M/\underline{\lambda}M) \cong L_R M \otimes_R R/\underline{\lambda}R$ in $\underline{\mathrm{CM}}(R/\underline{\lambda}R)$.*

Proof. Let $F_\bullet$ be an R-free resolution of M, then it is easy to see that $F_\bullet \otimes_R R/\underline{\lambda}R$ gives a free resolution of $M/\underline{\lambda}M$ over $R/\underline{\lambda}R$, because $\underline{\lambda}$ is also a regular sequence on M. Therefore, we have $\Omega^1_{R/\underline{\lambda}R}(M/\underline{\lambda}M) \cong \Omega_R^1 M \otimes_R R/\underline{\lambda}R$ in $\underline{\mathrm{CM}}(R/\underline{\lambda}R)$. It thus follows that

$$\begin{aligned} L_{R/\underline{\lambda}R}(M/\underline{\lambda}M) &\cong \mathrm{Hom}_{R/\underline{\lambda}R}(\Omega_R^1 M \otimes_R R/\underline{\lambda}R, R/\underline{\lambda}R) \\ &\cong \mathrm{Hom}_R(\Omega_R^1 M, R/\underline{\lambda}R) \cong L_R M \otimes_R R/\underline{\lambda}R. \qquad \square \end{aligned}$$

Let $\mathfrak{m} \supseteq \underline{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ be a regular sequences of length r. We denote the stable category of maximal Cohen–Macaulay modules over $R/\underline{\lambda}R$ by $\underline{\mathrm{CM}}(R/\underline{\lambda}R)$. We always consider the set of objects of $\underline{\mathrm{CM}}(R/\underline{\lambda}R)$ as a subset of the set of objects of $\underline{\mathrm{CM}}^r(R)$. Note that for two modules M_1 and M_2 in $\underline{\mathrm{CM}}(R/\underline{\lambda}R)$, $M_1 \cong M_2$ in the stable category $\underline{\mathrm{CM}}(R/\underline{\lambda}R)$ if and only if M_1 is stably isomorphic to M_2 in $R/\underline{\lambda}R$-mod, that is, there is an isomorphism $M_1 \oplus F \cong M_2 \oplus G$ as $R/\underline{\lambda}R$-modules for some free $R/\underline{\lambda}R$-modules F and G.

Definition 1.3 (*Linkage of Cohen–Macaulay modules*). Let N_1, N_2 be two Cohen–Macaulay modules of codimension r. We assume that N_1 (resp. N_2) is a maximal Cohen–Macaulay module over $R/\underline{\lambda}R$ (resp. $R/\underline{\mu}R$) for some regular sequence $\underline{\lambda}$ (resp. $\underline{\mu}$). If there exists a module $N \in \underline{\mathrm{CM}}^r(R)$ that belongs to both $\underline{\mathrm{CM}}(R/\underline{\lambda}R)$ and $\underline{\mathrm{CM}}(R/\underline{\mu}R)$ satisfying

$$N_1 \cong L_{R/\underline{\lambda}R}(N) \text{ in } \underline{\mathrm{CM}}(R/\underline{\lambda}R) \quad \& \quad N_2 \cong L_{R/\underline{\mu}R}(N) \text{ in } \underline{\mathrm{CM}}(R/\underline{\mu}R),$$

then we say N_1(resp. N_2) is linked to N through the regular sequence $\underline{\lambda}$ (resp. $\underline{\mu}$) and denote this by $N_1 \underset{\underline{\lambda}}{\sim} N$ (resp. $N_2 \underset{\underline{\mu}}{\sim} N$). We also say in this case that N_1 and N_2 are doubly linked through $(\underline{\lambda}, \underline{\mu})$, and denote it by $N_1 \underset{\underline{\lambda}}{\sim} N \underset{\underline{\mu}}{\sim} N_2$, or simply $N_1 \underset{(\underline{\lambda},\ \underline{\mu})}{\sim} N_2$.

If there is a sequence of modules $N_1, N_2, \ldots, N_s$ in $\underline{\mathrm{CM}}^r(R)$ such that N_i and N_{i+1} are doubly linked for $1 \le i < s$, then we say that N_1 and N_s are evenly linked.

Recalling the linkage of ideals from [6], we can see that the above definition agrees with it. Actually let $R \supseteq I$, J be Cohen–Macaulay ideals of codimension r and take a regular sequence $\underline{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ of length r contained in both I and J. Then I and J are linked through $\underline{\lambda}$ in the sense of [6] if and only if the Cohen–Macaulay modules R/I and R/J of codimension r are linked in the above sense (i.e. $R/I \underset{\underline{\lambda}}{\sim} R/J$). We remark that for $M \in \mathrm{CM}^r(\mathrm{R})$ we have $\Omega_R^r M \cong \Omega_R^r \Omega_R^{-r} \Omega_R^r(M) \cong \Omega_R^r \mathrm{X}_\mathrm{R}(\mathrm{M})$ in $\underline{\mathrm{CM}}(R)$. And we use this fact in the proofs of the following Theorem 1.4 and Corollary 1.5.

Theorem 1.4. *For a given regular sequence $\underline{\lambda}$ of length r in $\mathfrak{m}$, the following diagram commutes*:

$$\begin{array}{ccc}
\underline{\mathrm{CM}}(R/\underline{\lambda}R) & \xrightarrow{\mathrm{X}_R} & \underline{\mathrm{CM}}(R) \\
\Big\downarrow {\scriptstyle L_{R/\underline{\lambda}R}} & & \Big\downarrow {\scriptstyle L_R \circ \Omega_R^r} \\
\underline{\mathrm{CM}}(R/\underline{\lambda}R) & \xrightarrow[\mathrm{X}_R]{} & \underline{\mathrm{CM}}(R).
\end{array}$$

Proof. We shall prove the theorem after showing two claims. First we claim:

(i) *If N is in $\underline{\mathrm{CM}}(R/\underline{\lambda}R)$, then $D_R(\Omega_R^r N) \cong \mathrm{X}_R(D_{R/\underline{\lambda}R} N)$ in $\underline{\mathrm{CM}}(R)$.*

To show this note that $\mathrm{Ext}_R^i(N,R) = 0$ $(i \ne r)$ and $\mathrm{Ext}_R^r(N,R) \cong D_{R/\underline{\lambda}R}(N)$, since $N \in \underline{\mathrm{CM}}^r(R)$. Therefore from the long exact sequence

$$0 \to \Omega_R^r(N) \to F_{r-1} \to \cdots \to F_0 \to N \to 0$$

with F_i being R-free, we obtain the exact sequence

$$0 \to D_R(F_0) \to \cdots \to D_R(F_{r-1}) \to D_R(\Omega_R^r N) \xrightarrow{p} D_{R/\underline{\lambda}R}(N) \to 0.$$

This shows that the kernel of p is of finite projective dimension. Note here that $D_R(\Omega_R^r N) \in \underline{\mathrm{CM}}(R)$, and hence it follows from the definition that $D_R(\Omega_R^r N) \cong \mathrm{X}_R(D_{R/\underline{\lambda}R} N)$ in $\underline{\mathrm{CM}}(R)$.

Next we claim:

(ii) *If N is in $R/\underline{\lambda}R$-mod, then we have an isomorphism in* $\underline{\mathrm{CM}}(R)$:

$$\Omega_R^{r+1}(N) \cong \Omega_R^r(\Omega^1_{R/\underline{\lambda}R}(N)).$$

To show this, consider the free cover of N over $R/\underline{\lambda}R$ to have the short exact sequence:

$$0 \to \Omega^1_{R/\underline{\lambda}R}N \to G \to N \to 0,$$

where G is a free $R/\underline{\lambda}R$-module. Since $R/\underline{\lambda}R$ has projective dimension r as an R-module, taking the rth syzygies as R-modules we have the exact sequence (up to R-free summands)

$$0 \to \Omega_R^r(\Omega^1_{R/\underline{\lambda}R}N) \to F \to \Omega_R^r N \to 0,$$

where F is a free R-module. The isomorphism in (ii) results from this.

Utilizing the isomorpshims (i) and (ii), for $N \in \underline{\mathrm{CM}}(R/\underline{\lambda}R)$ we have the following isomorphisms in $\underline{\mathrm{CM}}(R)$:

$$\begin{aligned} \mathrm{X}_R(L_{R/\underline{\lambda}R}N) &\cong \mathrm{X}_R(D_{R/\underline{\lambda}R}\Omega^1_{R/\underline{\lambda}R}N) \cong D_R(\Omega_R^r\Omega^1_{R/\underline{\lambda}R}N) \\ &\cong D_R(\Omega_R^{r+1}N) \cong D_R(\Omega_R^{r+1}\mathrm{X}_R N) \cong L_R(\Omega_R^r \mathrm{X}_R N). \end{aligned}$$

This completes the proof of the theorem. □

Corollary 1.5. *Let $\{\underline{\lambda}, \underline{\mu}\} \subseteq \mathfrak{m}$ be a regular sequence of length $r+s$ where $\underline{\lambda}$ is of length r and $\underline{\mu}$ is of length s. Putting $R' = R/\underline{\lambda}R$ and $R'' = R/(\underline{\lambda}, \underline{\mu})R$, we have the following commutative diagram*:

$$\begin{tikzcd}
\underline{\mathrm{CM}}(R'') \arrow[r, "\Omega_R^{s+r}"] \arrow[d, equal] & \underline{\mathrm{CM}}(R) \arrow[r, equal] & \underline{\mathrm{CM}}(R) \arrow[d, equal] \\
\underline{\mathrm{CM}}(R'') \arrow[r, "\Omega_{R'}^{s}"] \arrow[d, "L_{R''}"'] & \underline{\mathrm{CM}}(R') \arrow[r, "\Omega_R^{r}"] \arrow[d, "L_{R'}"'] & \underline{\mathrm{CM}}(R) \arrow[d, "L_R"'] \\
\underline{\mathrm{CM}}(R'') \arrow[r, "\mathrm{X}_{R'}"] \arrow[d, equal] & \underline{\mathrm{CM}}(R') \arrow[r, "\mathrm{X}_R"] & \underline{\mathrm{CM}}(R) \arrow[d, equal] \\
\underline{\mathrm{CM}}(R'') \arrow[r, "\mathrm{X}_R"] & \underline{\mathrm{CM}}(R) \arrow[r, equal] & \underline{\mathrm{CM}}(R).
\end{tikzcd}$$

Proof. The commutativity of the top square of the diagram is easily obtained by the subsequent use of the second claim (ii) in the proof of Theorem 1.4.

For $M \in \underline{\mathrm{CM}}(R')$, we have the isomorphisms

$$L_R(\Omega_R^r(M)) \cong L_R(\Omega_R^r \mathrm{X}_R(M)),$$

which is isomorphic to $\mathrm{X}_R(L_{R'}(M))$ by Theorem 1.4. This shows the right square in the middle line commutes. Finally, to show the commutativity of the bottom square, we note from the theorem that

$$\mathrm{X}_R \mathrm{X}_{R'} L_{R''}(N) \cong L_R \Omega_R^r \Omega_{R'}^s(N) \cong L_R \Omega_R^{r+s}(N) \cong \mathrm{X}_R L_{R''}(N),$$

for any N in $\underline{\mathrm{CM}}(R'')$. Since $L_{R''}^2 \cong \mathrm{id}_{\underline{\mathrm{CM}}(R'')}$, substituting $L_{R''}(N)$ for N in the above equations, we obtain $\mathrm{X}_R(\mathrm{X}_{R'}(N)) \cong \mathrm{X}_R(N)$ in $\underline{\mathrm{CM}}(R)$. $\square$

Corollary 1.6. *Let N_1 and N_2 be modules in $\underline{\mathrm{CM}}^r(R)$. If N_1 and N_2 are doubly linked, then we have $\mathrm{X}_R(N_1) \cong \mathrm{X}_R(N_2)$ in $\underline{\mathrm{CM}}(R)$.*

Proof. If $N_1 \underset{\underline{\lambda}}{\sim} N \underset{\underline{\mu}}{\sim} N_2$ for some $N \in \underline{\mathrm{CM}}(R/\underline{\lambda}R) \cap \underline{\mathrm{CM}}(R/\underline{\mu}R)$, then it follows from the definition that $N_1 \cong L_{R/\underline{\lambda}R}(N)$ and $N_2 \cong L_{R/\underline{\mu}R}(N)$. Therefore applying Theorem 1.4, we get $\mathrm{X}_R(N_1) \cong L_R(\Omega_R^r N)$ and $\mathrm{X}_R(N_2) \cong L_R(\Omega_R^r N)$ in $\underline{\mathrm{CM}}(R)$. Hence $\mathrm{X}_R(N_1) \cong \mathrm{X}_R(N_2)$ as desired. $\square$

It turns out from Corollary 1.6 that the Cohen–Macaulay approximation functor X_R yields a map from the set of even linkage classes in $\underline{\mathrm{CM}}^r(R)$ to the set of objects in $\underline{\mathrm{CM}}(R)$.

Definition 1.7. Let us denote by $\mathrm{B}_r(R)$ the set of even linkage classes of modules in $\underline{\mathrm{CM}}^r(R)$. Then we can define a map Φ_r from $\mathrm{B}_r(R)$ to the set of isomorphism classes of modules in $\underline{\mathrm{CM}}(R)$ by $[N] \longmapsto \mathrm{X}_R(N)$.

2. A condition making the map Φ_2 surjective

If R is a local Gorenstein domain, then every Cohen–Macaulay module $M \in \mathrm{CM}(R)$ has a well-defined rank, say s, and a free module of rank s can be embedded in M:

$$0 \to R^s \to M \to N \to 0 \text{ (exact)},$$

where one can easily see that $N \in \mathrm{CM}^1(R)$. Hence taking a nonzero divisor x that annihilates N, we see that $N \in \underline{\mathrm{CM}}(R/xR)$ and that $M \cong \mathrm{X}_R(N)$. In this way, if R is a domain, then any maximal Cohen–Macaulay module over R is in the image of X_R from $\underline{\mathrm{CM}}^1(R)$, hence Φ_1 is a surjective map.

This argument can be slightly generalized in the following way using the theorem of Bourbaki.

Lemma 2.1. *Let R be a normal Gorenstein domain and let $M \in \underline{\mathrm{CM}}(R)$. For any integer $j \geq 1$, there is an ideal I of R such that $M \cong \Omega^{j+1}(R/I)$ in $\underline{\mathrm{CM}}(R)$.*

Proof. Taking the jth cosyzygy N of M, we have the exact sequence:

$$0 \to M \to F_{j-1} \to \cdots \to F_0 \to N \to 0.$$

Since $N \in \underline{\mathrm{CM}}(R)$ hence it is torsion free, it follows from Bourbaki's theorem [3, Section 4, Theorem 6] that there is an ideal I of R and an exact sequence:

$$0 \to R^{s-1} \to N \to I \to 0,$$

where s is the rank of N. Then by the above we have $M \cong \Omega^j N \cong \Omega^j I$, since $j \geq 1$. □

In this lemma the codimension of the module R/I is at most two. In the case that $R/I \in \underline{\mathrm{CM}}^r(R)$ and $\Omega^r(R/I) \cong M$, we see that $\mathrm{X}_R(R/I) \cong \Omega_R^{-r}(M)$, and hence $\Omega_R^{-r}(M)$ is the image of the even linkage class of R/I under the map Φ_r defined in (1.7).

As to the problem asking when a module in $\underline{\mathrm{CM}}(R)$ is the image of $\mathrm{B}_2(R)$ under the map Φ_2, we can show the following result.

Theorem 2.2. *Let R be a normal Gorenstein complete local ring of dimension 2. Then the following conditions are equivalent*:

(a) *R is a UFD.*

(b) *For any module $M \in \underline{\mathrm{CM}}(R)$, we can find an R-module L of finite length (hence a CM module of codimension 2) such that $M \cong \Omega_R^2(L)$ in $\underline{\mathrm{CM}}(R)$.*

(c) *The map Φ_2 is surjective onto the set of isomorphism classes of modules in $\underline{\mathrm{CM}}(R)$.*

Proof. (a) $\Rightarrow$ (b): For $M \in \underline{\mathrm{CM}}(R)$, let s be the rank of M and, embedding M into the free module R^s, we have the exact sequence:

$$0 \to M \to R^s \to N \to 0.$$

We can see that $\dim N = \operatorname{depth} N = 1$. Let $\{p_1, \ldots, p_n\}$ be all the associated prime ideals of N and set $S = R - \bigcup_{i=1}^n p_i$. Note that each p_i is a prime ideal of height one, and hence $S^{-1}R$ is a PID. Therefore there is an isomorphism of $S^{-1}R$-modules:

$$\phi : S^{-1}N \xrightarrow{\cong} \bigoplus_{i=1}^{s} S^{-1}R/a_i S^{-1}R$$

for some $a_i \in S^{-1}R$. Since $a_i S^{-1}R \cap R$ is an ideal of pure height one, and since R is a UFD, we can find an element $b_i \in R$ such that $b_i R = a_i S^{-1}R \cap R$. Taking b_i instead of a_i, we may assume that $a_i \in R$ and that every associated prime of a_i is one of the p_i. Now we can take a map of R-modules $f : N \to \bigoplus_{i=1}^{s} R/a_iR$ so that $S^{-1}f = \phi$ is an isomorphism. Here we claim that f is a monomorphism. In fact, if not, then taking an associated prime p of $\mathrm{Ker}(f)$, we see that p is associated to N hence is one of the p_i. But then f_p is an isomorphism which contradicts that $\mathrm{Ker}(f)_p \neq 0$. Now we set $L = \mathrm{Coker}(f)$:

$$0 \to N \xrightarrow{f} \bigoplus_{i=1}^{s} R/a_iR \to L \to 0.$$

Since $\mathrm{pd}_R R/a_i = 1$, we easily see that $\Omega_R^2(L) \cong \Omega_R^1(N) \cong M$ in $\underline{\mathrm{CM}}(R)$. Therefore, it remains to show that L is of finite length. By the definition we have

$$\mathrm{Supp}(L) \subseteq \mathrm{Supp}\left(\bigoplus_{i=1}^{s} R/a_iR\right) \subseteq \bigcup_{i=1}^{n} \mathrm{Supp}(R/p_i).$$

However f_{p_i} is an isomorphism for $1 \le i \le n$, and we can conclude that $\mathrm{Supp}(L)$ consists of only a closed point, hence L is of finite length.

The equivalence (b) $\Leftrightarrow$ (c) is obvious, because Φ_2 sends the even linkage class of $L \in \underline{\mathrm{CM}}^2(R)$ to $\Omega_R^{-2}\Omega_R^2(L) \in \underline{\mathrm{CM}}(R)$ and Ω_R^{-2} is an automorphism on the category $\underline{\mathrm{CM}}(R)$.

(b) $\Rightarrow$ (a): Let $\mathrm{Cl}(R)$ be the divisor class group of R and let p be an arbitrary prime ideal of R of height one. We have only to show that the class $c(p)$ in $\mathrm{Cl}(R)$ is trivial. Since $p \in \underline{\mathrm{CM}}(R)$, it follows from the condition (b) we have an R-module L of finite length such that $p \cong \Omega_R^2(L)$ in $\underline{\mathrm{CM}}(R)$. Then we have the following exact sequence:

$$0 \to p \oplus R^l \to R^m \to R^n \to L \to 0,$$

for some integers l, m and n. Now taking the divisor classes attatched to modules as in [3, Section 4, No. 7] (or the first Chern class), we have from this sequence that $c(p) = c(L) = 0$ in $\mathrm{Cl}(R)$, since L is of finite length. See [3, Section 4, Proposition 16]. $\square$

Example 2.3. Let k be a field and set $R = k[\![x, y, z]\!]/(x^2 - yz)$ that is a normal Gorenstein domain of dimension 2. Now let p be the ideal of R generated by $\{x, y\}$. It is easily verified that p is a prime ideal of height one and $\mathrm{Cl}(R)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ generated by $c(p)$. It is also known that p is a unique indecomposable non-free maximal Cohen–Macaulay module over R. By the proof of the above theorem, p is not in the image of Φ_2. On the other hand we can easily verify that $\Omega_R^2(k) \cong p \oplus p$. Therefore, we conclude that the image of Φ_2 is just the set of classes of modules that are isomorphic to the direct sum of even number of copies of p.

3. Linkage of CM modules of codimension 1

We have defined a map Φ_r in Definition (1.7) for any $r \geq 1$. In the case $r = 1$, the following proposition shows the condition for two classes in $B_1(R)$ to have the same image under Φ_1.

Proposition 3.1. *Let λ, μ be regular elements in $\mathfrak{m}$ and put $\xi = \lambda\mu$. And let N_1 (resp. N_2) be a module in $\underline{\mathrm{CM}}(R/\lambda R)$ (resp. $\underline{\mathrm{CM}}(R/\mu R)$). Then the following two conditions are equivalent*:

(a) $\mathrm{X}_R(N_1) \cong \mathrm{X}_R(N_2)$ *in* $\underline{\mathrm{CM}}(R)$.

(b) *There exists a module $N \in \underline{\mathrm{CM}}(R/\xi^2 R)$ that contains N_2 as a submodule such that* $\mathrm{pd}_R(N/N_2) < \infty$ *and* $N_1 \underset{(\xi,\xi^2)}{\sim} N$.

Proof. (a) $\Rightarrow$ (b): First note that $\Omega_R^1 N_1 \cong \Omega_R^1 N_2$ in $\underline{\mathrm{CM}}(R)$, since $\mathrm{X}_R(N_1) \cong \mathrm{X}_R(N_2)$. Now consider the R-free covers to get the exact sequences $0 \to K_1 \xrightarrow{\alpha} F_1 \to N_1 \to 0$ and $0 \to K_2 \xrightarrow{\beta} F_2 \to N_2 \to 0$. Since $K_1 \cong K_2$ in $\underline{\mathrm{CM}}(R)$, after adding a suitable free summand if necessary, we may assume that $K := K_1 = K_2$ and $F := F_1 = F_2$. Putting $N = \mathrm{Coker}(\xi\alpha)$, we first claim that $N_1 \underset{(\xi,\xi^2)}{\sim} N$. To show this, since $\xi N_1 = 0$, there exists a unique map $E_\xi(\alpha)$ that makes the following diagram commutative:

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \Omega^1_{R/\xi R} N_1 \arrow[d] & \\
0 \arrow[r] & F \arrow[r, "\xi"] \arrow[d, "E_\xi(\alpha)"'] & F \arrow[r] \arrow[d, equal] & F/\xi F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r, "\alpha"] \arrow[d] & F \arrow[r] & N_1 \arrow[r] \arrow[d] & 0. \\
 & \mathrm{Coker}(E_\xi(\alpha)) \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}
$$

It follows from this that $\mathrm{Coker}(E_\xi(\alpha)) \cong \Omega^1_{R/\xi R} N_1$. Applying the dual functor D_R, we thus get the isomorphisms in $\underline{\mathrm{CM}}(R/\xi R)$:

$$\begin{aligned} \mathrm{Coker}(D_R(E_\xi(\alpha))) &\cong D_{R/\xi R}(\mathrm{Coker}(E_\xi(\alpha))) \\ &\cong D_{R/\xi R}(\Omega^1_{R/\xi R} N_1) = L_{R/\xi R}(N_1). \end{aligned} \tag{1}$$

On the other hand, we also have the following commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & \Omega^1_{R/\xi^2 R} N_1 \arrow[d] & \\
0 \arrow[r] & F \arrow[r, "\xi^2"] \arrow[d, "E_\xi(\alpha)"'] & F \arrow[r] \arrow[d, equal] & F/\xi^2 F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r, "\xi\alpha"] \arrow[d] & F \arrow[r] & N \arrow[r] \arrow[d] & 0. \\
 & \mathrm{Coker}(E_\xi(\alpha)) \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

Thus we have the isomorphisms in $\underline{\mathrm{CM}}(R/\xi^2)$:

$$\begin{aligned} \mathrm{Coker}(D_R(E_\xi(\alpha))) &\cong D_{R/\xi^2 R}(\mathrm{Coker}(E_\xi(\alpha))) \\ &\cong D_{R/\xi^2 R}(\Omega^1_{R/\xi^2 R} N) = L_{R/\xi^2 R}(N). \end{aligned} \tag{2}$$

Combining the above isomorphisms in (1) and (2), we conclude that $N_1 \underset{(\xi,\xi^2)}{\sim} N$. It remains to show that N contains N_2 and $\mathrm{pd}_R(N/N_2) < \infty$. To prove this, note that there exists a unique map $E_\xi(\beta): F \to K$ such that $\beta \circ E_\xi(\beta) = \xi \mathrm{id}_F$, since $\xi N_2 = 0$. We can easily verify that $E_\xi(\beta) \circ \beta = \xi \mathrm{id}_K$ holds and hence that $\alpha \circ E_\xi(\beta) \circ \beta = \xi\, \alpha$. Thus we have the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r, "\beta"] \arrow[d, equal] & F \arrow[r] \arrow[d, "\alpha \circ E_{\xi}(\beta)"] & N_2 \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & K \arrow[r, "\xi \alpha"] & F \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & & \operatorname{Coker} \alpha \circ E_{\xi}(\beta) \arrow[r, equal] \arrow[d] & \operatorname{Coker} \psi \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

It follows from this diagram that we can find a monomorphism $\psi : N_2 \to N$ with $\mathrm{pd}_R(\operatorname{Coker}(\psi)) \leq 1$ as desired.

(b) $\Rightarrow$ (a): Since $\mathrm{pd}_R(N/N_2) < \infty$, noting $\Omega_R^n(N) \cong \Omega_R^n(N_2)$ for large n, we see that $\mathrm{X}_R(N_2) \cong \mathrm{X}_R(N)$ in $\underline{\mathrm{CM}}(R)$. On the other hand, since N and N_1 are doubly linked, we have $\mathrm{X}_R(N) \cong \mathrm{X}_R(N_1)$ in $\underline{\mathrm{CM}}(R)$ by Corollary 1.6. Hence $\mathrm{X}_R(N_1) \cong \mathrm{X}_R(N_2)$ in $\underline{\mathrm{CM}}(R)$. □

4. Linkage of CM modules over hypersurface rings

In this section we consider the following three hypersurface rings:

$$R = k[\![\underline{x}]\!]/(f),$$

$$R^\sharp = k[\![\underline{x}, y]\!]/(f + y^2),$$

$$R^{\sharp\sharp} = k[\![\underline{x}, y, z]\!]/(f + y^2 + z^2) \cong k[\![\underline{x}, u, v]\!]/(f + uv),$$

where $\underline{x} = \{x_1, \ldots, x_{d-1}\}, y, z$ are $d + 1$ variables over an algebraically closed field k of characteristic 0 where $d \geq 2$, and $u = y + \sqrt{-1}z$, $v = y - \sqrt{-1}z$ and f is a non zero element in $k[\![\underline{x}]\!]$. Note that $\{y\}$ (resp. $\{y, z\}$) is a regular sequence on $R^\sharp$ (resp. $R^{\sharp\sharp}$) and that $R \cong R^\sharp / yR^\sharp \cong R^{\sharp\sharp}/(y, z)R^{\sharp\sharp}$. Therefore, an object in $\underline{\mathrm{CM}}(R)$ can be naturally regarded as an object in $\underline{\mathrm{CM}}^1(R^\sharp)$ and $\underline{\mathrm{CM}}^2(R^{\sharp\sharp})$.

Let (φ_M, ψ_M) be a matrix factorization for $M \in \underline{\mathrm{CM}}(R)$, which is, by definition, a pair of two square matrices with entries in $k[\![\underline{x}]\!]$ satisfying $\varphi_M \circ \psi_M = \psi_M \circ \varphi_M = f \cdot 1$ and $\mathrm{Coker}\, \varphi_M \cong M$. Recalling Knörrer's periodicity theorem from [5], the functor $\mathrm{Lif} : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R^{\sharp\sharp})$ defined by

$$M \mapsto \mathrm{Coker} \begin{pmatrix} \varphi_M & u \cdot 1 \\ -v \cdot 1 & \psi_M \end{pmatrix}$$

gives the category equivalence. See [8, Ch. 12] for more details. Also recall that $\Omega^1_R M \cong \mathrm{Coker}\, \psi_M$ and $D_R M \cong \mathrm{Coker}\, {}^t\varphi_M$, and hence that $\Omega^2_R M \cong M$ and $L_R M \cong \mathrm{Coker}\, {}^t\psi_M$. These observations show the following:

Proposition 4.1. *The following diagram is commutative*:

$$\begin{array}{ccc} \underline{\mathrm{CM}}(R) & \xrightarrow{\mathrm{Lif}} & \underline{\mathrm{CM}}(R^{\sharp\sharp}) \\ {\scriptstyle L_R \circ \Omega^1_R} \big\downarrow & & \big\downarrow {\scriptstyle L_{R^{\sharp\sharp}}} \\ \underline{\mathrm{CM}}(R) & \xrightarrow{\mathrm{Lif}} & \underline{\mathrm{CM}}(R^{\sharp\sharp}). \end{array}$$

Proof. For a module M in $\underline{\mathrm{CM}}(R)$ we have the following isomorphisms in $\underline{\mathrm{CM}}(R^{\sharp\sharp})$ which proves the proposition.

$$\begin{aligned} L_{R^{\sharp\sharp}}(\mathrm{Lif}(M)) &\cong L_{R^{\sharp\sharp}} \left(\mathrm{Coker} \begin{pmatrix} \varphi_M & u \cdot 1 \\ -v \cdot 1 & \psi_M \end{pmatrix} \right) \cong \mathrm{Coker} \begin{pmatrix} {}^t\psi_M & v \cdot 1 \\ -u \cdot 1 & {}^t\varphi_M \end{pmatrix} \\ &\cong \mathrm{Coker} \begin{pmatrix} {}^t\varphi_M & u \cdot 1 \\ -v \cdot 1 & {}^t\psi_M \end{pmatrix} \cong \mathrm{Lif}(\mathrm{Coker}\, {}^t\varphi_M) \\ &\cong \mathrm{Lif}(D_R M) \cong \mathrm{Lif}(L_R \Omega^1_R M). \qquad \square \end{aligned}$$

Lemma 4.2. $\Omega^2_{R^{\sharp\sharp}} M \cong \mathrm{Lif}(M \oplus \Omega^1_R M) \cong \Omega^n_{R^{\sharp\sharp}} M$ *for* $M \in \underline{\mathrm{CM}}(R)$ *and for any integer* $n \geq 2$.

Proof. We quote $\Omega^1_{R^{\sharp\sharp}} \Omega^1_{R^\sharp} M \cong \mathrm{Lif}\,(M) \oplus \Omega^1_{R^{\sharp\sharp}}\, \mathrm{Lif}\,(M)$ from [8, the proof of Theorem 12.10, the last line, p. 115]. This implies $\Omega^2_{R^{\sharp\sharp}} M \cong \mathrm{Lif}(M \oplus \Omega^1_R M)$, since we have $\Omega^1_{R^{\sharp\sharp}} \Omega^1_{R^\sharp} M \cong \Omega^2_{R^{\sharp\sharp}} M$ from Corollary 1.5 and we can easily check that $\Omega^1_{R^{\sharp\sharp}} \mathrm{Lif}(M) \cong \mathrm{Lif}(\Omega^1_R M)$. Therefore we have

$$\begin{aligned} \Omega^3_{R^{\sharp\sharp}} M &\cong \Omega^1_{R^{\sharp\sharp}} \Omega^2_{R^{\sharp\sharp}} M \cong \Omega^1_{R^{\sharp\sharp}} \mathrm{Lif}(M \oplus \Omega^1_R M) \\ &\cong \mathrm{Lif}(\Omega^1_R M \oplus \Omega^2_R M) \cong \mathrm{Lif}(\Omega^1_R M \oplus M) \cong \Omega^2_{R^{\sharp\sharp}} M. \qquad \square \end{aligned}$$

Lemma 4.3. $\Omega^1_{R^\sharp} M \cong \Omega^n_{R^\sharp} M \cong \Omega^1_{R^\sharp} \Omega^1_R M$ *for* $M \in \underline{\mathrm{CM}}(R)$ *and any integer* $n \geq 1$.

Proof. Let (φ_M, ψ_M) be a matrix factorization for M. It then follows from [8, Lemma 12.3, p. 108] that a matrix factorization of $\Omega^1_{R^\sharp}M$ is given by

$$\left(\begin{pmatrix} \psi_M & -y\cdot 1 \\ y\cdot 1 & \varphi_M \end{pmatrix}, \begin{pmatrix} \varphi_M & y\cdot 1 \\ -y\cdot 1 & \psi_M \end{pmatrix}\right).$$

Hence we have

$$\Omega^1_{R^\sharp}M \cong \mathrm{Coker}\begin{pmatrix} \psi_M & -y\cdot 1 \\ y\cdot 1 & \varphi_M \end{pmatrix} \cong \mathrm{Coker}\begin{pmatrix} \varphi_M & y\cdot 1 \\ -y\cdot 1 & \psi_M \end{pmatrix} \cong \Omega^2_{R^\sharp}M.$$

Thus by induction on n, we get $\Omega^1_{R^\sharp}M \simeq \Omega^n_{R^\sharp}M$. On the other hand, we have shown in Corollary 1.5 that $\Omega^1_{R^\sharp}\Omega^1_R M \cong \Omega^2_{R^\sharp}M$. $\square$

Proposition 4.4. *The following conditions are equivalent for M_1 and M_2 in $\underline{\mathrm{CM}}(R)$:*

(a) $M_1 \oplus \Omega^1_R M_1 \cong M_2 \oplus \Omega^1_R M_2$ *in* $\underline{\mathrm{CM}}(R)$.

(b) $\mathrm{X}_{R^\sharp}(M_1) \cong \mathrm{X}_{R^\sharp}(M_2)$ *in* $\underline{\mathrm{CM}}(R^\sharp)$.

(c) $\mathrm{X}_{R^{\sharp\sharp}}(M_1) \cong \mathrm{X}_{R^{\sharp\sharp}}(M_2)$ *in* $\underline{\mathrm{CM}}(R^{\sharp\sharp})$.

Proof. (a)$\Rightarrow$(b): Note that $\Omega^1_{R^\sharp}M_i \cong \Omega^1_{R^\sharp}\Omega^1_R M_i$ for $i=1,2$ by Lemma 4.3. We thus obtain

$$\Omega^1_{R^\sharp}M_1 \oplus \Omega^1_{R^\sharp}M_1 \cong \Omega^1_{R^\sharp}M_2 \oplus \Omega^1_{R^\sharp}M_2$$

by applying the functor $\Omega^1_{R^\sharp}$ to the both side of (a). This implies $\Omega^1_{R^\sharp}M_1 \cong \Omega^1_{R^\sharp}M_2$, since R is complete local ring and hence the Kull–Schmit theorem holds for finitely generated modules.

(b)$\Rightarrow$(c): Just apply the functor $\mathrm{X}_{R^{\sharp\sharp}}$ to the both sides of (b) and we get (c) because of Corollary (1.5).

(c)$\Rightarrow$(a): Since $\Omega^2_{R^{\sharp\sharp}}M_1 \simeq \Omega^2_{R^{\sharp\sharp}}M_2$, we obtain from Lemma 4.2

$$\mathrm{Lif}(M_1 \oplus \Omega^1_R M_1) \cong \mathrm{Lif}(M_2 \oplus \Omega^1_R M_2).$$

This implies (a) since the functor Lif is fully faithful. $\square$

Corollary 4.5. *Let M_1 and M_2 be in $\underline{\mathrm{CM}}(R)$. Suppose that they belong to the same even linkage class in $\underline{\mathrm{CM}}^1(R^\sharp)$ or in $\underline{\mathrm{CM}}^2(R^{\sharp\sharp})$. Then we have $M_1 \oplus \Omega_R M_1 \cong M_2 \oplus \Omega_R M_2$. Furthermore, if we assume that both modules are indecomposable, then we must have either $M_1 \cong M_2$ or $M_1 \cong \Omega^1_R M_2$.*

Example 4.6. Using this corollary we are sometimes able to find the condition for given modules to belong to the same even linkage class.

For the simplest example, let $R^\sharp = k[\![x,y]\!]/(x^n+y^2)$ and $R = k[\![x]\!]/(x^n)$. Take an integer r as $n=2r$ or $2r+1$. It is easy to see that the set of classes of indecomposable modules in $\underline{\mathrm{CM}}(R)$ is $\{R/(x^i) \mid 1 \le i < n\}$.

Then we can claim that the modules $R/(x^i)$ for $1 \le i \le r$ define r distinct even linkage classes in $\underline{\mathrm{CM}}(R^\sharp)$.

In fact, if $R/(x^i)$ and $R/(x^j)$ belong to the same even linkage class in $\underline{\mathrm{CM}}^1(R^\sharp)$, then, since $\Omega_R(R/(x^i)) \cong R/(x^{n-i})$, it follows from the corollary that $R/(x^i) \cong R/(x^j)$ or $R/(x^i) \cong R/(x^{n-j})$, but since we assumed $1 \le i,\ j \le r$, we must have $i = j$.

Note that $R/(x^i)$ and $R/(x^{n-i})$ belong to the same even linkage class in $\underline{\mathrm{CM}}^1(R^\sharp)$ for $1 \le i \le r$. This is just a result of computation as follows:

$$R/(x^i) \underset{x^{i+1}y}{\sim} R^\sharp/(x^{i+1}y, x^{n+1}) \underset{x^{n+1}}{\sim} R/(x^{n-i}).$$

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